

**Sixth Term Examination Papers**

**MATHEMATICS 1**

**MONDAY 23 JUNE 2008**

**9465**

Afternoon

Time: 3 hours

Additional materials: Answer paper  
Graph paper  
Formulae booklet

**Candidates may not use electronic calculators**



**INSTRUCTIONS TO CANDIDATES**

**Please read this page carefully, but do not open this question paper until you are told that you may do so.**

Write your name, Centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

**INFORMATION FOR CANDIDATES**

Each question is marked out of 20. There is no restriction of choice.

You will be assessed on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with Mathematical Formulae and Tables.

**Electronic calculators are not permitted.**

**Please wait to be told you may begin before turning this page.**

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**This question paper consists of 6 printed pages and 2 blank pages.**

## Section A: Pure Mathematics

- 1** What does it mean to say that a number  $x$  is *irrational*?

Prove by contradiction statements A and B below, where  $p$  and  $q$  are real numbers.

**A:** If  $pq$  is irrational, then at least one of  $p$  and  $q$  is irrational.

**B:** If  $p + q$  is irrational, then at least one of  $p$  and  $q$  is irrational.

Disprove by means of a counterexample statement C below, where  $p$  and  $q$  are real numbers.

**C:** If  $p$  and  $q$  are irrational, then  $p + q$  is irrational.

If the numbers  $e$ ,  $\pi$ ,  $\pi^2$ ,  $e^2$  and  $e\pi$  are irrational, prove that at most one of the numbers  $\pi + e$ ,  $\pi - e$ ,  $\pi^2 - e^2$ ,  $\pi^2 + e^2$  is rational.

- 2** The variables  $t$  and  $x$  are related by  $t = x + \sqrt{x^2 + 2bx + c}$ , where  $b$  and  $c$  are constants and  $b^2 < c$ . Show that

$$\frac{dx}{dt} = \frac{t - x}{t + b},$$

and hence integrate  $\frac{1}{\sqrt{x^2 + 2bx + c}}$ .

Verify by direct integration that your result holds also in the case  $b^2 = c$  if  $x + b > 0$  but that your result does not hold in the case  $b^2 = c$  if  $x + b < 0$ .

3 Prove that, if  $c \geq a$  and  $d \geq b$ , then

$$ab + cd \geq bc + ad. \quad (*)$$

(i) If  $x \geq y$ , use (\*) to show that  $x^2 + y^2 \geq 2xy$ .

If, further,  $x \geq z$  and  $y \geq z$ , use (\*) to show that  $z^2 + xy \geq xz + yz$  and deduce that  $x^2 + y^2 + z^2 \geq xy + yz + zx$ .

Prove that the inequality  $x^2 + y^2 + z^2 \geq xy + yz + zx$  holds for all  $x, y$  and  $z$ .

(ii) Show similarly that the inequality

$$\frac{s}{t} + \frac{t}{r} + \frac{r}{s} \geq 3$$

holds for all positive  $r, s$  and  $t$ .

[**Note:** The final part of this question differs (though not substantially) from what appeared in the actual examination since this was found to be unsatisfactory (though not incorrect) in a way that had not been anticipated.]

4 A function  $f(x)$  is said to be *convex* in the interval  $a < x < b$  if  $f''(x) \geq 0$  for all  $x$  in this interval.

(i) Sketch on the same axes the graphs of  $y = \frac{2}{3} \cos^2 x$  and  $y = \sin x$  in the interval  $0 \leq x \leq 2\pi$ .

The function  $f(x)$  is defined for  $0 < x < 2\pi$  by

$$f(x) = e^{\frac{2}{3} \sin x}.$$

Determine the intervals in which  $f(x)$  is convex.

(ii) The function  $g(x)$  is defined for  $0 < x < \frac{1}{2}\pi$  by

$$g(x) = e^{-k \tan x}.$$

If  $k = \sin 2\alpha$  and  $0 < \alpha < \pi/4$ , show that  $g(x)$  is convex in the interval  $0 < x < \alpha$ , and give one other interval in which  $g(x)$  is convex.

5 The polynomial  $p(x)$  is given by

$$p(x) = x^n + \sum_{r=0}^{n-1} a_r x^r,$$

where  $a_0, a_1, \dots, a_{n-1}$  are fixed real numbers and  $n \geq 1$ . Let  $M$  be the greatest value of  $|p(x)|$  for  $|x| \leq 1$ . Then *Chebyshev's theorem* states that  $M \geq 2^{1-n}$ .

(i) Prove Chebyshev's theorem in the case  $n = 1$  and verify that Chebyshev's theorem holds in the following cases:

(a)  $p(x) = x^2 - \frac{1}{2}$ ;

(b)  $p(x) = x^3 - x$ .

(ii) Use Chebyshev's theorem to show that the curve  $y = 64x^5 + 25x^4 - 66x^3 - 24x^2 + 3x + 1$  has at least one turning point in the interval  $-1 \leq x \leq 1$ .

6 The function  $f$  is defined by

$$f(x) = \frac{e^x - 1}{e - 1}, \quad x \geq 0,$$

and the function  $g$  is the inverse function to  $f$ , so that  $g(f(x)) = x$ . Sketch  $f(x)$  and  $g(x)$  on the same axes.

Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} f(x) dx + \int_0^k g(x) dx = \frac{1}{2(\sqrt{e} + 1)},$$

where  $k = \frac{1}{\sqrt{e} + 1}$ , and explain this result by means of a diagram.

7 The point  $P$  has coordinates  $(x, y)$  with respect to the origin  $O$ . By writing  $x = r \cos \theta$  and  $y = r \sin \theta$ , or otherwise, show that, if the line  $OP$  is rotated by  $60^\circ$  clockwise about  $O$ , the new  $y$ -coordinate of  $P$  is  $\frac{1}{2}(y - \sqrt{3}x)$ . What is the new  $y$ -coordinate in the case of an anti-clockwise rotation by  $60^\circ$ ?

An equilateral triangle  $OBC$  has vertices at  $O$ ,  $(1, 0)$  and  $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ , respectively. The point  $P$  has coordinates  $(x, y)$ . The perpendicular distance from  $P$  to the line through  $C$  and  $O$  is  $h_1$ ; the perpendicular distance from  $P$  to the line through  $O$  and  $B$  is  $h_2$ ; and the perpendicular distance from  $P$  to the line through  $B$  and  $C$  is  $h_3$ .

Show that  $h_1 = \frac{1}{2}|y - \sqrt{3}x|$  and find expressions for  $h_2$  and  $h_3$ .

Show that  $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$  if and only if  $P$  lies on or in the triangle  $OBC$ .

- 8 (i) The gradient  $y'$  of a curve at a point  $(x, y)$  satisfies

$$(y')^2 - xy' + y = 0. \quad (*)$$

By differentiating  $(*)$  with respect to  $x$ , show that either  $y'' = 0$  or  $2y' = x$ .

Hence show that the curve is either a straight line of the form  $y = mx + c$ , where  $c = -m^2$ , or the parabola  $4y = x^2$ .

- (ii) The gradient  $y'$  of a curve at a point  $(x, y)$  satisfies

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0.$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

## Section B: Mechanics

- 9** Two identical particles  $P$  and  $Q$ , each of mass  $m$ , are attached to the ends of a diameter of a light thin circular hoop of radius  $a$ . The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially,  $P$  is in contact with the table. At time  $t$ , the hoop has rotated through an angle  $\theta$ . Write down the position at time  $t$  of  $P$ , relative to its starting point, in cartesian coordinates, and determine its speed in terms of  $a$ ,  $\theta$  and  $\dot{\theta}$ . Show that the total kinetic energy of the two particles is  $2ma^2\dot{\theta}^2$ .

Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

- 10** On the (flat) planet Zog, the acceleration due to gravity is  $g$  up to height  $h$  above the surface and  $g'$  at greater heights. A particle is projected from the surface at speed  $V$  and at an angle  $\alpha$  to the surface, where  $V^2 \sin^2 \alpha > 2gh$ . Sketch, on the same axes, the trajectories in the cases  $g' = g$  and  $g' < g$ .

Show that the particle lands a distance  $d$  from the point of projection given by

$$d = \left( \frac{V - V'}{g} + \frac{V'}{g'} \right) V \sin 2\alpha,$$

where  $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}$ .

- 11** A straight uniform rod has mass  $m$ . Its ends  $P_1$  and  $P_2$  are attached to small light rings that are constrained to move on a rough rigid circular wire with centre  $O$  fixed in a vertical plane, and the angle  $P_1OP_2$  is a right angle. The rod rests with  $P_1$  lower than  $P_2$ , and with both ends lower than  $O$ . The coefficient of friction between each of the rings and the wire is  $\mu$ . Given that the rod is in limiting equilibrium (i.e. on the point of slipping at both ends), show that

$$\tan \alpha = \frac{1 - 2\mu - \mu^2}{1 + 2\mu - \mu^2},$$

where  $\alpha$  is the angle between  $P_1O$  and the vertical ( $0 < \alpha < 45^\circ$ ).

Let  $\theta$  be the acute angle between the rod and the horizontal. Show that  $\theta = 2\lambda$ , where  $\lambda$  is defined by  $\tan \lambda = \mu$  and  $0 < \lambda < 22.5^\circ$ .

## Section C: Probability and Statistics

**12** In this question, you may use without proof the results:

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1) \quad \text{and} \quad \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1).$$

The independent random variables  $X_1$  and  $X_2$  each take values  $1, 2, \dots, N$ , each value being equally likely. The random variable  $X$  is defined by

$$X = \begin{cases} X_1 & \text{if } X_1 \geq X_2 \\ X_2 & \text{if } X_2 \geq X_1. \end{cases}$$

- (i) Show that  $P(X = r) = \frac{2r-1}{N^2}$  for  $r = 1, 2, \dots, N$ .
- (ii) Find an expression for the expectation,  $\mu$ , of  $X$  and show that  $\mu = 67.165$  in the case  $N = 100$ .
- (iii) The median,  $m$ , of  $X$  is defined to be the integer such that  $P(X \geq m) \geq \frac{1}{2}$  and  $P(X \leq m) \geq \frac{1}{2}$ . Find an expression for  $m$  in terms of  $N$  and give an explicit value for  $m$  in the case  $N = 100$ .
- (iv) Show that when  $N$  is very large,

$$\frac{\mu}{m} \approx \frac{2\sqrt{2}}{3}.$$

**13** Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

- (i) Show that the probability that each husband sits next to his wife is  $\frac{2}{15}$ .
- (ii) Find the probability that exactly two husbands sit next to their wives.
- (iii) Find the probability that no husband sits next to his wife.

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